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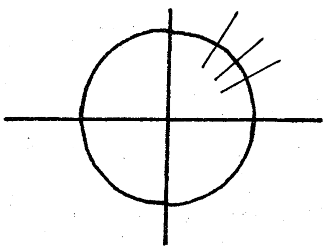
# TRANSVERSALITY OF LINEAR HOLOMORPHIC VECTOR FIELD ON $\mathbb{C}^n$

TOSHIKAZU ITO *
( Ryukoku University )

## INTRODUCTION

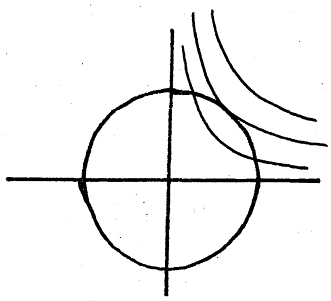
Let $\mathcal{F}$ be the holomorphic foliation on $\mathbb{C}^2 - \{0\}$ defined by a linear vector field $X = \sum_{i=1}^{2} \lambda_i z_i \, \partial/\partial z_i$ ( $\lambda_i \in \mathbb{C}$, $\lambda_i \neq 0$ ) on $\mathbb{C}^2$. Let us begin by recalling a well-known fact ( [1],[2] ) :

FACT If $\lambda_1/\lambda_2$ does not belong to $\mathbb{R}_- = \{$ negative real numbers $\}$, then the 3 dimensional unit sphere $S^3$ in $\mathbb{C}^2$ is transverse to $\mathcal{F}$. On the other hand, if $\lambda_1/\lambda_2 \in \mathbb{R}_-$, $S^3$ is not transverse to $\mathcal{F}$.



$\lambda_1/\lambda_2 \notin \mathbb{R}_-$

Fig. 1



$\lambda_1/\lambda_2 \in \mathbb{R}_-$

Fig. 2

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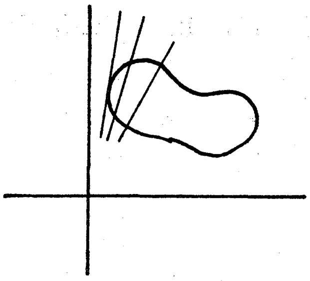
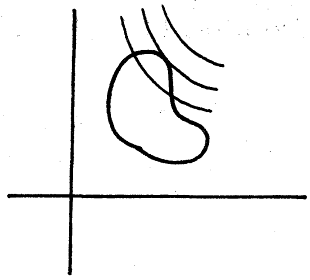


$\lambda_1/\lambda_2 \notin \mathbf{R}_-$

Fig. 3



$\lambda_1/\lambda_2 \in \mathbf{R}_-$

Fig. 4

These pictures suggest to us a question :

QUESTION Let M be a connected closed 2 or 3 dimensional smooth manifold. Is there a smooth map $\phi$ ; $M \longrightarrow \mathbf{C}^2 - \{0\}$ such that $\phi$ is transverse to $\mathfrak{F}$ ?

In this note, we shall give an answer to this question. This note is divided into three sections. In §1, we give a definition of transversality of maps to a holomorphic vector field on $\mathbf{C}^n$ and some examples. In §2, we investigate a non-existence of transversal maps. Finally in §3, we have a structure theorem on an existence of transverse maps on condition that $\lambda_1/\lambda_2$ is a real positive number.

The author would like to thank Professor Andre Haefliger for numerous discussions and encouragement.

## § 1 DEFINITION OF TRANSVERSALITY OF MAPS TO A HOLOMORPHIC VECTOR FIELD ON $\mathbf{C}^n$ AND WELL-KNOWN EXAMPLES

Let $\mathcal{F}$ be the holomorphic foliation on $\mathbf{C}^n$ defiened by the solutions of a holomorphic vector field X on $\mathbf{C}^n$. Let M be a smooth manifold of dimension 2n-2 or 2n-1 and $\phi$ a smooth map from M in $\mathbf{C}^n$.

DEFINITION 1.1 We say that the map $\phi$ ; $M \longrightarrow \mathbf{C}^n$ is transverse to $\mathcal{F}$ if the following identity satisfies for all points $p \in M$ :

$$\phi_*( T_p(M) ) + T_{\phi(p)}(\mathcal{F}) = T_{\phi(p)}( \mathbf{C}^n )$$

We shall here give the well-known examples.

EXAMPLE 1.2 ( [1],[2] ) Let $X = \sum_{i=1}^{n} \lambda_i z_i \, \partial/\partial z_i$ a linear holomorphic vector field on $\mathbf{C}^n$. We assume that $\lambda_i \notin \mathbf{R}\lambda_j$ for $i \neq j$. The convex hull of $\Lambda = \{ \lambda_1, \ldots , \lambda_n \}$ in $\mathbf{C}$ is denoted $\mathcal{H}( \Lambda )$.

(i) If the origin 0 in $\mathbf{C}$ belongs to $\mathcal{H}( \Lambda )$, then the 2n-1 dimensional unit sphere $S^{2n-1}$ in $\mathbf{C}^n$ is not transverse to $\mathcal{F}$.

(ii) If the origin 0 in $\mathbf{C}$ does not belong to $\mathcal{H}( \Lambda )$, then $S^{2n-1}$ is transverse to $\mathcal{F}$.

## § 2 NON-EXISTENCE OF TRANSVERSE MAPS

First, the Fig.3 and 4 lead us to a theorem : Poincaré-

Hopf type theorem for holomorphic vector field.

THEOREM 2.1 ( [3] ) Let $\mathcal{F}$ be the holomorphic foliation on $\mathbf{C}^n$ defined by the solutions of a holomorphic vector field $X$ on $\mathbf{C}^n$, $n \geqq 2$. If a smooth imbedding $\phi$ from the $2n-1$ dimensional sphere $S^{2n-1}$ in $\mathbf{C}^n$ is transverse to $\mathcal{F}$, then $X$ has only one singular point in the inside of $\phi(S^{2n-1})$.

Secondly, in the special case of a linear vector field $X = \sum_{i=1}^{n} \lambda_i z_i \, \partial/\partial z_i$ on $\mathbf{C}^n$, $n \geqq 2$, we have some results.

COROLLARY 2.2 ( [5] ) Let $\mathcal{F}$ be the foliation on $\mathbf{C}^n - \{0\}$ defined by $X = \sum_{i=1}^{n} \lambda_i z_i \, \partial/\partial z_i$ ( $\lambda_i \neq 0$, $i = 1, \ldots, n$ ). A smooth imbedding $\phi$ ; $S^{2n-1} \longrightarrow \mathbf{C}^n - \{0\}$ which is homotope to zero in $\Pi_{2n-1}(\mathbf{C}^n - \{0\})$ is not transverse to $\mathcal{F}$.

COROLLARY 2.3 ( [5] ) Let $\mathcal{F}$ be the foliation on $\mathbf{C}^2 - \{0\}$ defined by $X = \sum_{i=1}^{2} \lambda_i z_i \, \partial/\partial z_i$ ( $\lambda_i \neq 0$, $i = 1,2$ ). Let $T^3$ be the torus of dimension 3. A smooth imbedding $\phi$ ; $T^3 \longrightarrow \mathbf{C}^2 - \{0\}$ which satisfies the following property (#) is not transverse to $\mathcal{F}$.

Property (#) : There exists a smooth imbedding $\Phi$ ; $S^1 \times S^1 \times D^2 \longrightarrow \mathbf{C}^2$ such that $\Phi$ satisfies $\Phi|_{\partial(S^1 \times S^1 \times D^2)} = \phi$ and the image of $\Phi$ contains zero in $\mathbf{C}^2$.

THEOREM 2.4 ( [5] ) Let $\mathcal{F}$ be the foliation on $\mathbf{C}^n - \{0\}$ defined by $X = \sum_{i=1}^{n} \lambda_i z_i \, \partial/\partial z_i$ ( $\lambda_i \neq 0$, $i = 1, \ldots, n$ ).

Assume that at least one of $\lambda_i/\lambda_j$ ( $i \neq j$ ) is a negative real number. Let M be a closed connected smooth manifold of dimension 2n-2 or 2n-1. Then there is not a smooth map $\phi$ , $M \longrightarrow \mathbf{C}^n - \{0\}$ which is transverse to $\mathcal{F}$.

THEOREM 2.5 ( [4] ) Let $\mathcal{F}$ be the foliation on $\mathbf{C}^n - \{0\}$ defined by $X = \sum_{i=1}^{n} \lambda_i z_i \, \partial/\partial z_i$ ( $\lambda_i \neq 0$, $i = 1, \ldots , n$ ). Assume that $\lambda_i \notin R\lambda_j$ ( $i \neq j$ ) and $0 \in \mathcal{H}(\Lambda)$. Let M be a closed connected smooth manifold of dimension 2n-1. Then there is not a smooth map $\phi$ ; $M \longrightarrow C^n - \{0\}$ which is transverse to $\mathcal{F}$ .

## § 3 EXISTENCE OF TRANSVERSE MAP AND STRUCTURE THEOREMS

Let $\mathcal{F}$ be the holomorphic foliation on $\mathbf{C}^n - \{0\}$ defined by a linear vector field $X = \sum_{i=1}^{n} \lambda_i z_i \, \partial/\partial z_i$ ( $\lambda_i \neq 0$, $i = 1, \ldots , n$ ). Assume that all $\lambda_i/\lambda_j$ ( $i \neq j$ ) are positive rational numbers. Then the 2n-1 dimensional unit sphere $S^{2n-1}$ in $\mathbf{C}^n$ is transverse to $\mathcal{F}$ and the foliation on $S^{2n-1}$ defined by $\mathcal{F}$ is a generalized Seifert structure. Now, we have a structure theorem.

THEOREM 3.1 ( [5] ) Let $\mathcal{F}$ be the foliation on $\mathbf{C}^n - \{0\}$ defined by $X = \sum_{i=1}^{n} \lambda_i z_i \, \partial/\partial z_i$ ( $\lambda_i \neq 0$, $i = 1, \ldots , n$ ). Assume that all $\lambda_i/\lambda_j$ ( $i \neq j$ ) are positive real numbers. Let M be a closed connected smooth manifold of dimension 2n-1. If a smooth map $\phi$ ; $M \longrightarrow \mathbf{C}^n - \{0\}$ is transverse to $\mathcal{F}$, then

$M$ is diffeomorphic to the sphere $S^{2n-1}$ of dimension $2n-1$.

Because of the existence of 2-field on a manifold $M$ which is transverse to $\mathcal{F}$ defined by a holomorphic vector field $X$ on $\mathbf{C}^{2n}$, $n \geqq 1$, we have other structure theorem.

THEOREM 3.2 ( [5] ) Let $\mathcal{F}$ be the foliation on $\mathbf{C}^2$ defined by a holomorphic vector field $X$ on $\mathbf{C}^2$. Let $M$ be a closed connected smooth manifold of dimension 2. If a smooth map $\phi$ ; $M \longrightarrow \mathbf{C}^2$ is transverse to $\mathcal{F}$, then $M$ is diffeomorphic to the torus $T^2$. Moreover, in the case of a linear vector field $X = \sum_{i=1}^{2} \lambda_i z_i \,\partial/\partial z_i$ ( $\lambda_i \neq 0$, $i = 1,2$ and $\lambda_1/\lambda_2 \notin \mathbf{R}$ ), we can construct a smooth map $\phi$ ; $T^2 \longrightarrow \mathbf{C}^2$ such that $\phi$ is transverse to $\mathcal{F}$.

Finally, we shall give some examples of transverse maps.

EXAMPLE 3.3 ( [5] ) Let $\mathcal{F}$ be the foliation on $\mathbf{C}^2 - \{0\}$ defined by $X = \sum_{i=1}^{2} \lambda_i z_i \,\partial/\partial z_i$ ( $\lambda_i \neq 0$, $i=1,2$ and $\lambda_1/\lambda_2 \notin \mathbf{R}$ ) Let $M$ be $T^3$ or $S^2 \times S^1$. Then we can construct a smooth map $\phi$ ; $M \longrightarrow \mathbf{C}^2 - \{0\}$ such that $\phi$ is transverse to $\mathcal{F}$.
( cf. §2 Corollary 2.3 )

EXAMPLE 3.4 ( [4] ) Let $\mathcal{F}$ be the foliation on $\mathbf{C}^n - \{0\}$ defined by $X = \sum_{i=1}^{n} \lambda_i z_i \,\partial/\partial z_i$ ( $\lambda_i \neq 0$, $i = 1, \ldots, n$ ). Assume that $\lambda_i \notin R\lambda_j$ ( $i \neq j$ ) and $0 \notin \mathcal{H}(\Lambda)$. then there exists a smooth imbedding $\phi$ ; $S^1 \times S^{2n-3} \times S^1 \longrightarrow \mathbf{C}^n - \{0\}$ which is transverse to $\mathcal{F}$.

# REFERENCES

[1] V.I.Arnold : Chapitres supplémentaires de la théorie des équations différentielles ordinaires. Editions Mir Moscou (1980).

[2] C.Camacho, N.H.Kuiper and J.Palis : The topology of holomorphic flows with singularity. Publ. Math. I.H.E.S., 48 (1978), 5-38.

[3] T.Ito : Poincaré-Hopf type theorem for holomorphic vector field. J. of Toyota College of Technology 21 (1988), 109-112.

[4] T.Ito : Transversality of linear holomrphic vector field on $\mathbf{C}^n$. Stability theory and related topics in dynamical systems (1989), 73- 79.

[5] T.Ito : Existence et non-existence d'applications variété fermée de dimension deux ou trois dans $\mathbf{C}^2 - \{0\}$ transverse à un champ de vecteurs linéaires. in preparation